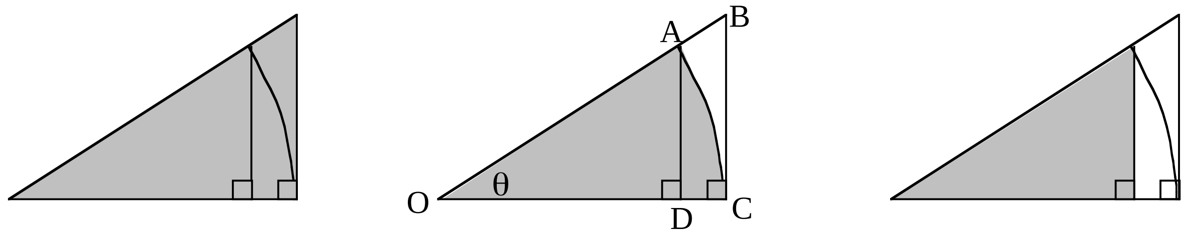
Proof of $\lim_{x \to 0} \frac{\sin x}{x}$



Area 1 Area 2 Area 3

Let circle O be a unit circle, so that the length of OC = 1

$$\text{Area 1} > \text{Area 2} > \text{Area 3} \qquad \text{(obvious from the diagram)}$$

$$\frac{1}{2}bh > \frac{\theta}{2\pi}\left(r^2\pi\right) > \frac{1}{2}bh \qquad \text{(area formulas)}$$

$$\frac{1}{2}(1)(\tan\theta) > \frac{\theta}{2\pi}(\pi) > \frac{1}{2}(\cos\theta)(\sin\theta) \qquad \text{(substitutions)}$$

$$\frac{\tan\theta}{2} > \frac{\theta}{2} > \frac{\cos\theta \cdot \sin\theta}{2} \qquad \text{(simplify)}$$

$$\tan\theta > \theta > \cos\theta \cdot \sin\theta \qquad \text{(multiply by 2)}$$

$$\frac{\sin\theta}{\cos\theta} > \theta > \cos\theta \cdot \sin\theta \qquad \text{(trig identity)}$$

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > \cos\theta \qquad (\div \sin\theta)$$

$$\sec\theta > \frac{\theta}{\sin\theta} > \cos\theta \qquad \text{(trig identity)}$$

$\lim_{\theta \to 0} \sec\theta = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$, so by the sandwich theorem

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$