# Proof of the derivative of sin x

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \quad \text{(definition)}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h} \quad \text{(identity for the sine of a sum)}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h-\sin x+\sin h\cos x}{h} \quad \text{(rearrange terms)}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h} \quad \text{(factor out sin x)}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\sin h\cos x}{h}$$
(limit of a sum is the sum of the limits)

$$= \lim_{h\to 0}\sin x\frac{(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h} \quad \text{(rearrange)}$$

$$= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$
(limit of a constant times a function)

$$= \sin x\,(0)+\cos x\,(1) \quad \text{(evaluation of limits)}$$

$$= 0+\cos x \quad \text{(arithmetic)}$$

$$= \cos x$$