Proof of the derivative of ln x

$$\frac{d}{dx}\ln x = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \qquad \text{(definition)}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\ln(x+h)-\ln x\right) \qquad \text{(rearrange terms)}$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \qquad \text{(law of logarithms)}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{x}{x}\ln\left(\frac{x+h}{x}\right) \qquad \text{(multiply by a clever form of 1)}$$

$$= \lim_{h\to 0}\frac{1}{x}\frac{x}{h}\ln\left(1+\frac{h}{x}\right) \qquad \text{(rearrange terms)}$$

$$= \lim_{h\to 0}\frac{1}{x}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \qquad \text{(law of logarithms)}$$

$$= \frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \qquad \text{(property of limits)}$$

$$= \frac{1}{x}\ln e \qquad \text{(evaluation of limit)}$$

$$= \frac{1}{x}\cdot 1 \qquad \text{(evaluation of the logarithm)}$$

$$= \frac{1}{x} \qquad \text{(arithmetic)}$$