

Proof of $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \frac{\cos x + 1}{\cos x + 1} \quad (\text{multiply by a clever form of one})$$

$$= \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \quad (\text{distribute})$$

$$= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x(\cos x + 1)} \quad (\text{Pythagorean trig identity})$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{-\sin x}{\cos x + 1} \quad (\text{limit of a product is the product of limits})$$

$$= 1 \cdot \frac{0}{2} \quad (\text{evaluate limits})$$

$$= 0 \quad (\text{arithmetic})$$