2007 BC Calculus Free Response Solutions

1 a) $A = 2\int_0^3 \left(\frac{20}{1+x^2} - 2\right)dx \approx 37.962$

b) $V = 2\pi\int_0^3 \left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right)dx \approx 1871.190$

c) $V = \frac{\pi}{4}\int_0^3 \left(\frac{20}{1+x^2} - 2\right)^2 dx \approx 174.268$

2 a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0$, 3, and 7.
At $t = 0$, gallons = 5000.
At $t = 3$, gallons = $5000 + \int_0^3 f(t)\,dt - 250(3) \approx 5126.591$
At $t = 7$, gallons = $5126.591 + \int_3^7 f(t)\,dt - 2000(4) \approx 4513.807$
The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

3 a) $A = \frac{2}{3}(2)^2\pi + 2\int_{2\pi/3}^{\pi} \frac{1}{2}(3+2\cos\theta)^2\,d\theta \approx 10.370$

b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} \approx -1.732$

Because $\frac{dr}{d\theta}$ is the rate of change of r with respect to θ, this means the particle is getting closer to the origin when $\theta = \frac{\pi}{3}$.

c) $y = r\sin\theta = (3+2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (3+2\cos\theta)\cos\theta + \sin\theta(-2\sin\theta)$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} \approx 0.5$

Because $\frac{dy}{dt}$ is the rate of change of y with respect to t, this means the particle is getting further away from the x-axis when $\theta = \frac{\pi}{3}$.

4 a) $m = f'(e) = e^2$

$y - 2 = e^2(x - e)$

b) $f''(x) = x(1 + 2\ln x)$

For $1 < x < 3$, $\ln x > 1$, and $x > 0$, so both factors of $f''(x)$ are positive so $f''(x)$ is positive and $f(x)$ is concave up on $(1, 3)$.

c) Integration by parts yields:

$$f(x) = \int f'(x)\,dx = \int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

$$f(e) = = \frac{e^3}{3} - \frac{e^3}{9} + C \quad \text{so} \quad C = 2 - \frac{2}{9}e^3$$

$$\therefore f(x) = \frac{x^3}{3}\ln x - \frac{x^3}{9} + 2 - \frac{2}{9}e^3$$

5 a) $r - 30 = 2(t - 5)$

$@t = 5.4,\ r = 30.8\text{ ft}$

Since the graph of r is concave down the tangent line lies above the curve, and this estimate is too large.

b) $\left.\frac{dV}{dt}\right|_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$

c) $\int_0^{12} r'(t)\,dt \approx 19.3\text{ft}$ $\quad \int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.

d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation is less than $\int_0^{12} r'(t)\,dt$.

6 a) Using the substitution principle:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \ldots + \frac{(-1)^n x^{2n}}{n!} + \ldots$$

b) $$\frac{1 - x^2 - f(x)}{x^4} = \frac{-\frac{x^4}{2} + \frac{x^6}{3!} + \ldots}{x^4} = -\frac{1}{2} + \frac{x^2}{3!} + \ldots$$

Thus, $\lim_{x\to 0} \frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2}$

c) $$\int_0^x e^{-t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots$$

Using only the first two terms $\int_0^x e^{-t^2}\,dt = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$

d) The series is alternating, so the error in this approximation is no bigger than $\frac{\left(\frac{1}{2}\right)^5}{10}$, the next term in the series. This evaluates to $\frac{1}{320}$, which is less than $\frac{1}{200}$.